IX. MRA AND CONSTRUCTION OF WAVELETS (PART TWO)

In this chapter and next, we will discuss the problem of obtaining orthonormal wavelets from a general Multiresolution Analysis. For the definition of a Multiresolution Analysis, we refer the reader to Chapter 7, definition 1(or Chapter 5, definition 2). One of the important fact about Multiresolution Analysis $\{V_n\}_{n\in\mathbb{Z}}$ is that there is a function $\varphi \in V_0$ such that $\{\varphi(x-l)|l \in \mathbb{Z}\}$ is a complete orthonormal system for V_0. The function φ is called a **scaling function** for the Multiresolution Analysis $\{V_n\}_{n\in\mathbb{Z}}$. Other facts we will be using in these two chapters are that for each $n \in \mathbb{Z}$, $V_n \subset V_{n+1}$ and that for any $f \in L^2(\mathbb{R})$, $f(x) \in V_n \iff f(2x) \in V_{n+1}$.

As before, for each $n \in \mathbb{Z}$, we let $W_n = V_{n+1} \ominus V_n$,the orthogonal complement of V_n in V_{n+1}. Equivalently, $V_{n+1} = V_n \oplus W_n$. As we have seen in last chapter, if there is a function $\psi \in W_0$ such that $\{\psi(x-l)|l \in \mathbb{Z}\}$ is a completeorthonormal system in W_n, then ψ is a orthonormal wavelet in $L^2(\mathbb{R})$. We will show that for any Multiresolution Analysissuch, such function ψ always exists, and it can be explicitly expressed in terms of scaling function and it associated **low pass filter.**

In this chapter, we will develop certain important facts about functions whose integral translates form an orthonormal system in $L^2(\mathbb{R})$, and introduce the concept of Low pass filter, preparing for the construction of wavelets in next chapter. In these two chapters, we will always identify any function f defined on $[-\pi, \pi]$ satisfying

$$\int_{-\pi}^{\pi} |f(x)|^2 dx < \infty$$

with its 2π-periodic extension on $(-\infty, \infty)$. We still call the collection of these functions as $L^2(-\pi, \pi)$. According to Chapter 1, for any $f \in L^2(-\pi, \pi)$, we have $f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{int}$ where the convergence is in the norm $L^2(-\pi, \pi)$ and $\hat{f}(n) = \langle f, e^{int} \rangle_{L^2(-\pi,\pi)}$ with $\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 < \infty$. In other words, $\{\hat{f}(n)\}_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$. Conversely, for any $\{c_n\}_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$, there is a $g \in L^2(-\pi, \pi)$ such that $g(x) = \sum_{n\in\mathbb{Z}} c_n e^{int}$ where the convergence is in the norm $L^2(-\pi, \pi)$, namely, for any $\varepsilon > 0$, there is a natural number N, such that for any integer $n > N$, we have

$$||g(t) - \sum_{|l|\leq n} c_l e^{ilt}||_{L^2(-\pi,\pi)} < \varepsilon.$$

Typeset by $\mathcal{AMS}$-TEX

First let us study such function $g(x) \in L^2(\mathbb{R})$ that $\{g(x-l)|l \in \mathbb{Z}\}$ is an orthonormal system. To this end, we need some properties of Fourier transform on $L^2(\mathbb{R})$. Note that the properties in the following lemma are true for $f \in L^1(\mathbb{R})$. The extension of these properties to the case when $f \in L^2(\mathbb{R})$ is left for the reader.

Lemma 1. *a)For any $f \in L^2(\mathbb{R})$, if $g(x) = f(x-l)$ for some fixed real number l, then $\hat{g}(\xi) = e^{-il\xi}\hat{f}(\xi)$.*

b)For any $f \in L^2(\mathbb{R})$, if $h(x) = \lambda f(\lambda x)$ for some fixed real number $\lambda > 0$, then $\hat{h}(\xi) = \hat{f}(\frac{\xi}{\lambda})$.

We also need following two lemmas. We list them below without proving. The first one below is a special type of **Fubini's Theorem**. The range of the integration is intentionally keep vague. It could be any interval or $\mathbb{R}$ itself. The second one below is some important fact about Fourier series for functions in $L^1(-\pi,\pi)$ which we do not have time to get into in this course.

Lemma 2. *If $\sum_{k\in\mathbb{Z}} \int |f_k|dx < \infty$ (or $\int \sum_{k\in\mathbb{Z}} |f_k|dx < \infty$), then $\sum_{k\in\mathbb{Z}} \int f_k dx = \int \sum_{k\in\mathbb{Z}} f_k dx$.*

Lamma 3. *Let $f(x)$ be a 2π-periodic function with $\int_{-\pi}^{\pi} |f(x)|dx < \infty$. If $f(x)$ satisfies $\int_{-\pi}^{\pi} f(x)e^{int}dx = \delta_{n,0}$, then $f(x) \equiv 1$.*

The following theorem is a key step in our discussion. The method used in its proof is also significant.

Theorem 1. *If $g \in L^2(\mathbb{R})$, than the following two statements are equivalent:*

(a)$\{g(x-l)|l \in \mathbb{Z}\}$ is a orthonormal system in $L^2(\mathbb{R})$.

(b)$\sum_{k\in\mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2 = 1$ holds for any $\xi \in \mathbb{R}$.

Proof. First note that the statement (a) is equivalent to the statement that $\langle g(x-k), g(x-l)\rangle = \delta_{k,l}$ for any $k,l \in \mathbb{Z}$, which in turn is equivalent to the statement that $\langle g(x), g(x-l)\rangle = \delta_{l,0}$ for any $l \in \mathbb{Z}$.

We first prove the implication "(a)⇒(b)". Assuming that statement (a) is true. Then

$$1 = \langle g, g\rangle = \frac{1}{2\pi}\langle \hat{g}, \hat{g}\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{g}(\xi)|^2 d\xi = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \int_{2k\pi-\pi}^{2k\pi+\pi} |\hat{g}(\xi)|^2 d\xi$$

where we have used Parseval's Identity and rewrite the resulting integral as a series of integrals. Next we apply the change of variable to each integral above, we note that the resulting series of integrals satisfies the condition of Fubini's Theorem (Lemma 2), so we apply Lemma 2 to conclude the sequence of computation:

$$1 = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_{-\pi}^{\pi}|\hat{g}(\mu+2k\pi)|^2 d\mu = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\sum_{k\in\mathbb{Z}}|\hat{g}(\mu+2k\pi)|^2)d\mu.$$

If we denote $G(\mu) = \sum_{k\in\mathbb{Z}}|\hat{g}(\mu+2k\pi)|^2$, then clearly $G(\mu)$ is a 2π-periodic function satisfying

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|G(\mu)|d\mu = \frac{1}{2\pi}\int_{-\pi}^{\pi}G(\mu)d\mu = 1.$$

In view of Lemma 3, to show that $G(\mu)\equiv 1$, we only need to check that for any $l\in\mathbb{Z}\setminus\{0\}$, $\frac{1}{2\pi}\int_{-\pi}^{\pi}G(\mu)e^{il\mu}d\mu = 0$. Indeed, since $\{g(x-l)|l\in\mathbb{Z}\}$ is a orthonormal system,

$$0 = \langle g(x), g(x-l)\rangle = \frac{1}{2\pi}\langle\widehat{g(x)}, \widehat{g(x-l)}\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{g}(\xi)\overline{\hat{g}(\xi)e^{-il\xi}}d\xi$$

$$= \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_{2k\pi-\pi}^{2k\pi+\pi}|\hat{g}(\xi)|^2 e^{il\xi}d\xi = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_{-\pi}^{\pi}|\hat{g}(\mu+2k\pi)|^2 e^{il\xi}d\mu.$$

where we use Parseval's Identity first. We also use Lemma 1 to find Fourier transform of function $f(x-l)$. The resulting integral then is written as a series of integrals and change of variables to each integral is performed, note that the condition of Fubini's Theorem (Lemma 2) is also satisfied, so finally by applying Lemma 2, we have

$$0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\sum_{k\in\mathbb{Z}}|\hat{g}(\mu+2k\pi)|^2)e^{il\xi}d\mu = \frac{1}{2\pi}\int_{-\pi}^{\pi}G(\mu)e^{il\mu}d\mu,$$

according to Lemma 3, then $G(\mu) = \sum_{k\in\mathbb{Z}}|\hat{g}(\mu+2k\pi)|^2 \equiv 1$ so we are done.

Now we prove the implication "(b)⇒(a)". Assuming that statement (b) is true. Namely $G(\mu) = \sum_{k\in\mathbb{Z}}|\hat{g}(\mu+2k\pi)|^2 \equiv 1$. Then for any $l\in\mathbb{Z}$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}G(\mu)e^{i\mu l}d\mu = \delta_{l,0}.$$

Note that again Fubini's Theorem applies, so all the computations done in the proof of "(a)⇒(b)" can be reverted to get $\langle g(x), g(x-l)\rangle = \delta_{l,0}$ for any $l\in\mathbb{Z}$, which means that $\{g(x-l)|l\in\mathbb{Z}\}$ is a orthonormal system in $L^2(\mathbb{R})$. □

The function $g\in \text{Ł}^2(\mathbb{R})$ described in Theorem 1 also has the following important property:

Theorem 2. *Let $g \in L^2(\mathbb{R})$ be such a function that for any $\xi \in \mathbb{R}$, $\sum_{k\in\mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2 = 1$. Let $h(\xi) = \sum_{l\in\mathbb{Z}} c_l e^{il\xi} \in L^2(-\pi,\pi)$. Then $h(\xi)\hat{g}(\xi) \in L^2(\mathbb{R})$ and*

$$h(\xi)\hat{g}(\xi) = \sum_{l\in\mathbb{Z}} c_l \hat{g}(\xi) e^{il\xi}.$$

Remark From Chapter 1, we know that whenever $h(\xi) = \sum_{l\in\mathbb{Z}} c_l e^{il\xi} \in L^2(-\pi,\pi)$, then we always have $\{c_l\}_{l\in\mathbb{Z}} \in l^2(\mathbb{Z})$. On the other hand, whenever $\{c_l\}_{l\in\mathbb{Z}} \in l^2(\mathbb{Z})$, the function defined by $h(\xi) = \sum_{l\in\mathbb{Z}} c_l e^{il\xi}$ is always in $L^2(-\pi,\pi)$.

Proof. First we prove that $h(\xi)\hat{g}(\xi) \in L^2(\mathbb{R})$ by showing that $\int_{-\infty}^{\infty} |h(\xi)|^2 |\hat{g}(\xi)|^2 d\xi < \infty$. Note that the above integral can be written as a series of integrals and be subject to change of variables the same way as in the proof of last Theorem, so

$$\int_{-\infty}^{\infty} |h(\xi)|^2|\hat{g}(\xi)|^2 d\xi = \sum_{k\in\mathbb{Z}} \int_{2k\pi-\pi}^{2k\pi+\pi} |h(\xi)|^2|\hat{g}(\xi)|^2 d\xi = \sum_{k\in\mathbb{Z}} \int_{-\pi}^{\pi} |h(\mu)|^2|\hat{g}(\mu+2k\pi)|^2 d\mu$$

Note that

$$\int_{-\pi}^{\pi} |h(\mu)|^2 (\sum_{k\in\mathbb{Z}} |\hat{g}(\mu + 2k\pi)|^2) d\mu = \int_{-\pi}^{\pi} |h(\mu)|^2 d\mu = 2\pi ||h||^2_{L^2(-\pi,\pi)} < \infty$$

So the condition for Fubini's Theorem (Lemma 2) is satisfied, the last integral of first line of equalities and the first integral of second line of equalities are identical, hence

$$\int_{-\infty}^{\infty} |h(\xi)|^2|\hat{g}(\xi)|^2 d\xi = \int_{-\pi}^{\pi} |h(\mu)|^2 (\sum_{k\in\mathbb{Z}} |\hat{g}(\mu + 2k\pi)|^2) d\mu < \infty.$$

Next we prove $h(\xi)\hat{g}(\xi) = \sum_{l\in\mathbb{Z}} c_l \hat{g}(\xi) e^{il\xi}$. Consider the following

$$||h(\xi)\hat{g}(\xi) - \sum_{|l|\le N, l\in\mathbb{Z}} c_l \hat{g}(\xi) e^{il\xi}||_2^2 = ||\hat{g}(\xi)\big(h(\xi) - \sum_{|l|\le N, l\in\mathbb{Z}} c_l e^{il\xi}\big)||_2^2.$$

Through similar computation we can show that it is equal to

$$2\pi ||h(\xi) - \sum_{|l|\le N, l\in\mathbb{Z}} c_l e^{il\xi}||_2^2.$$

Detailed computation and proof using $\varepsilon - N$ language is left to the reader. □

For the introduction of low pass filter and constructions in the next Chapter, we also need a Lemma to allow us freely switch back and forth between time-domain and frequency-domain. Specifically, we need

Lemma 4. *If $\{\phi_n\}_{n\in\mathbb{Z}}$ is a orthonormal system in $L^2(\mathbb{R})$. Then for any $f \in L^2(\mathbb{R})$, the following are equivalent:*

(a) $f(x) = \sum_{n\in\mathbb{Z}} c_n\phi_n(x)$ where the convergence is in the norm of $L^2(\mathbb{R})$.

(b)$\hat{f}(\xi) = \sum_{n\in\mathbb{Z}} c_n\hat{\phi}_n(\xi)$ where the convergence is in the norm of $L^2(\mathbb{R})$.

Remark According to Chapter 1, when $f(x) = \sum_{n\in\mathbb{Z}} c_n\phi_n(x)$ (where the convergence is in the norm of $L^2(\mathbb{R})$), then $c_n = \langle f, \phi_n\rangle$ for any $n \in \mathbb{Z}$ and $\{c_n\}_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$. On the other hand , for any $\{c_n\}_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$, $\sum_{n\in\mathbb{Z}} c_n\phi_n(x)$ converges (in the norm of $L^2(\mathbb{R})$) to some function in $L^2(\mathbb{R})$.

Proof. From Parseval's Identity, we have, for each $N \in \mathbb{N}$,

$$||\hat{f}(\xi) - \sum_{|n|\le N} c_n\hat{\phi}_n(\xi)||_2^2 = ||f(x) - \sum_{|n|\le N} c_n\phi_n(x)||_2^2.$$

The rest of $\varepsilon - N$ details are left for the reader. □

Now we are ready to introduce the concept of **Low Pass Filter**. Given an arbitrary multiresolution Analysis $\{V_n\}_{n\in\mathbb{Z}}$, according to its definition, we know that there is a scaling function $\phi(x)$ in V_0 such that $\{\phi(x-l)|l \in \mathbb{Z}\}$ is a complete orthonormal system for V_0. Moreover, by definition,

$$\frac{1}{2}\phi(\frac{x}{2}) \in V_{-1} \subset V_0.$$

Hence by Lemma 4 of Chapter 1, there exists $\{a_l\}_{l\in\mathbb{Z}} \in l^2(\mathbb{Z})$ (in fact, $a_l = \langle\frac{1}{2}\phi(\frac{x}{2}), \phi(x+l)\rangle_{L^2(\mathbb{R})}$ for each $l \in \mathbb{Z}$, but we do not need the specifics at the moment) such that

$$\frac{1}{2}\phi(\frac{x}{2}) = \sum_{l\in\mathbb{Z}} a_l\phi(x+l)$$

where the convergence is in the norm of $L^2(\mathbb{R})$. Hence by Lemma 4 and Lemma 1, we have that

$$\hat{\phi}(2\xi) = \sum_{l\in\mathbb{Z}} a_l\hat{\phi}(\xi)e^{il\xi}$$

where the convergence is in the norm of $L^2(\mathbb{R})$. Note that since $\{a_l\}_{l\in\mathbb{Z}} \in l^2(\mathbb{Z})$, so $\sum_{l\in\mathbb{Z}} a_le^{il\xi}$ converges in the norm of $L^2(-\pi,\pi)$ and is a function in $L^2(-\pi,\pi)$. We denote $m_0(\xi) = \sum_{l\in\mathbb{Z}} a_le^{il\xi}$ and call it the **Low Pass Filter** induced by the multiresolution with ϕ as scaling function. Finally, by Theorem 2, we have

$$\hat{\phi}(2\xi) = (\sum_{l\in\mathbb{Z}} a_l e^{il\xi})\hat{\phi}(\xi).$$

Using the special notation for low pass filter, we can write the above as

$$\hat{\phi}(2\xi) = \hat{\phi}(\xi)m_0(\xi).$$

We quickly state an important property of the $L^2(-\pi,\pi)$ function m_0.

Theorem 3. *If m_0 is a low pass filter induced by the multiresolution analysis with ϕ as scaling function. Then*

$$|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 \equiv 1.$$

Proof. Since $\phi(x)$ is the scaling function, so $\{\phi(x-l)|l \in \mathbb{Z}\}$ is an orthonormal system in $L^2(\mathbb{R})$, according to Theorem 1, we have

$$\sum_{k\in\mathbb{Z}} |\hat{g}(\xi+2k\pi)|^2 \equiv 1$$

so certainly

$$\sum_{k\in\mathbb{Z}} |\hat{g}(2\xi+2k\pi)|^2 \equiv 1.$$

Since $\hat{\phi}(2\xi) = \hat{\phi}(\xi)m_0(\xi)$ for each $\xi \in \mathbb{R}$, we have

$$\sum_{k\in\mathbb{Z}} |\hat{g}(\xi+k\pi)|^2 |m_0(\xi+k\pi)|^2 \equiv 1.$$

So

$$1 \equiv \sum_{k=2l\in\mathbb{Z}} |\hat{g}(\xi+k\pi)|^2|m_0(\xi+k\pi)|^2 + \sum_{k=2l+1\in\mathbb{Z}} |\hat{g}(\xi+k\pi)|^2|m_0(\xi+k\pi)|^2$$

$$= \sum_{l\in\mathbb{Z}} |\hat{g}(\xi+2l\pi)|^2|m_0(\xi+2l\pi)|^2 + \sum_{l\in\mathbb{Z}} |\hat{g}(\xi+(2l+1)\pi)|^2|m_0(\xi+(2l+1)\pi)|^2.$$

Note that m_0 is a 2π-periodic function, so

$$|m_0(\xi+2l\pi)|^2 = |m_0(\xi)|^2,$$

$$|m_0(\xi+(2l+1)\pi)|^2 = |m_0(\xi+\pi)|^2$$

for any $l \in \mathbb{Z}$. Thus, by using Theorem 1 one more time, we obtain

$$1 \equiv |m_0(\xi)|^2 + |m_0(\xi+\pi)|^2.$$

□